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Geometric and Dynamical Structures in Euclidean Space with Torsion: A Metric-Affine Approach

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ABSTRACT
This paper investigates the geometric and dynamical properties of Euclidean space endowed with a torsion tensor that framework is known as metric-affine geometry with a flat metric. We analyze the consequences of relaxing the torsion-free condition in Euclidean geometry. While classical Euclidean geometry assumes a symmetric, metric-compatible connection (Levi-Civita), we allow the connection to have an antisymmetric part determined by an independent torsion tensor. This simple yet rich extension reveals profound departures from Riemannian intuition: geodesics may deviate from straight lines, parallel transport becomes path-dependent even in flat space, and curvature can emerge purely from torsion. We derive explicit formulas for the connection, curvature, and geodesic equations in Cartesian coordinates. Through variational principles, we obtain field equations that couple torsion to the geometry, illustrating how torsion can be interpreted as an independent physical field (e.g., electromagnetic). The results highlight the operational meaning of torsion through closure failure of infinitesimal parallelograms and provide a clear pedagogical foundation for more advanced theories such as Einstein–Cartan, teleparallel, and metric-affine gravity.

KEYWORDS
Metric-affine geometry; Torsion tensor; Euclidean space; Geodesics; Curvature; Field equations; Variational principle; Einstein–Cartan theory; Teleparallel gravity.
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1. Introduction

In standard differential geometry, Euclidean space is characterized by a flat metric $g_{ij} = \delta_{ij}$ and a symmetric, metric-compatible connection (the Levi-Civita connection), which vanishes in Cartesian coordinates. This framework underlies classical mechanics and field theories in flat spacetime. However, generalizing the connection to include *torsion*—a tensor S^k_{ij} antisymmetric in its lower indices—leads to a richer geometry known as *metric-affine geometry* (Cartan, 1922; Hehl et al., 1976; Trautman, 2006). Such geometries have gained attention in unified field theories, where torsion is often

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associated with spin densities or electromagnetic fields (Hammond, 2002; Shapiro, 2002; Puetzfeld, 2005).

The foundational paper by Yaremenko (2014) developed the general theory of spaces with geometric structure generated jointly by metric and torsion tensors, deriving field equations from a variational principle. Building upon this work, we explore the simplest non-trivial case: **Euclidean space with torsion**. This space, denoted by Y^n following Yaremenko's notation, possesses a flat Euclidean metric but is endowed with a non-zero torsion tensor. Despite the flat metric, the presence of torsion modifies parallel transport, geodesic motion, and even induces curvature.

Our objectives are:

(1) To derive explicit coordinate expressions for the connection, curvature, and geodesics in Euclidean Y^n with torsion, using Yaremenko's decomposition theorem.
(2) To demonstrate the geometric meaning of torsion via the failure of parallelogram closure and compute the associated gap in the Euclidean context.
(3) To obtain the field equations from a variational principle in this simplified setting and discuss their physical interpretation, particularly in relation to electromagnetism.
(4) To provide concrete examples, such as constant totally antisymmetric torsion in three dimensions, illustrating how torsion generates curvature in an otherwise flat Euclidean space.

This study serves as a pedagogical illustration of the effects of torsion in a simplified setting, clarifying the geometric and physical implications of torsion independently of metric curvature. The results provide a foundation for understanding more complex metric-affine geometries and their potential applications in unified field theories.

2. Geometric Structure of Euclidean Space with Torsion

2.1. *Metric and Torsion Tensors in Y^n*

Following [20], we denote by Y^n an n-dimensional differentiable manifold endowed with a geometric structure generated jointly by a metric tensor g_{ij} and a torsion tensor S_{ij}^k. In the general theory developed by [20], Y^n represents a space where both curvature and torsion can coexist, with the connection determined uniquely by the metric compatibility condition and the given torsion tensor.

In the present work, we focus on the special case where Y^n is a Euclidean space—that is, the manifold is $\mathbb{R}^n$ and the metric is flat. In Cartesian coordinates $\{x^i\}$, the metric takes the simple form $g_{ij} = \delta_{ij}$.

The torsion tensor S_{ij}^k is introduced as an independent geometric object that is antisymmetric in its lower indices:

$$S_{ij}^k = -S_{ji}^k. \tag{2.1}$$

The metric tensor defines lengths and angles, while the torsion tensor characterizes the antisymmetric part of the connection and, as we shall see, influences parallel transport and geodesic motion.

The connection Γ_{ij}^k of Y^n is required to be *metric-compatible*, meaning the covariant derivative of the metric vanishes:

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^l_{ik} g_{lj} - \Gamma^l_{jk} g_{il} = 0. \tag{2.2}$$

For the Euclidean metric $g_{ij} = \delta_{ij}$, this condition reduces to:

$$\Gamma_{ijk} + \Gamma_{jik} = 0, \tag{2.3}$$

where $\Gamma_{ijk} = \delta_{kl}\Gamma^l_{ij}$. Equation (2.3) expresses the fact that, in Cartesian coordinates, the connection coefficients are antisymmetric in the first two indices when all indices are lowered with the Euclidean metric. This property is crucial for the subsequent derivation of the connection in terms of the torsion tensor.

In the general case of a non-Euclidean metric, the connection is uniquely determined by the metric and torsion via the decomposition theorem of [20]. In the Euclidean setting, however, the connection will be shown to depend solely on the torsion tensor, illustrating how torsion can induce non-trivial geometric effects even in a flat metric background.

2.2. *Connection Decomposition Theorem*

Following [20, Theorem 1], given a metric tensor g_{ij} and a torsion tensor S^k_{ij} on the space Y^n, there exists a unique metric-compatible connection Γ^p_{kl}. This connection can be decomposed into two distinct geometric objects:

$$\Gamma^p_{kl} = P^p_{kl} + L^p_{kl}, \tag{2.4}$$

where P^p_{kl} depends only on the metric and its derivatives, while L^p_{kl} incorporates both metric and torsion. Explicitly:

$$P^p_{kl} = \frac{1}{2} g^{pi}(g_{ik,l} + g_{li,k} - g_{kl,i}), \tag{2.5}$$

$$L^p_{kl} = \frac{1}{2} S^p_{kl} + \frac{1}{2} g^{pi}(g_{km} S^m_{li} + g_{lm} S^m_{ki}). \tag{2.6}$$

The object P^p_{kl} has the same form as the Christoffel symbols in Riemannian geometry but does not transform as a connection under coordinate changes when torsion is present. The tensor L^p_{kl} consists of two parts: the antisymmetric torsion tensor and a symmetric combination that couples the metric to torsion.

In the special case of Euclidean Y^n with metric $g_{ij} = \delta_{ij}$ in Cartesian coordinates, all partial derivatives of the metric vanish ($g_{ij,k} = 0$), and the inverse metric is simply $g^{ij} = \delta^{ij}$. This leads to significant simplifications:

$$P^p_{kl} = 0, \tag{2.7}$$

$$L^p_{kl} = \frac{1}{2}\left(S^p_{kl} + S^k_{lp} + S^l_{kp}\right), \tag{2.8}$$

where indices are raised and lowered with the Kronecker delta δ_{ij}. Consequently, the connection reduces to:

$$\Gamma^p_{kl} = \frac{1}{2}\left(S^p_{kl} + S^k_{lp} + S^l_{kp}\right). \tag{2.9}$$

A direct calculation confirms that the antisymmetric part of this connection equals the torsion tensor:

$$\Gamma^p_{kl} - \Gamma^p_{lk} = S^p_{kl}, \tag{2.10}$$

which is consistent with the definition of torsion as the antisymmetric part of the connection. Equation (2.9) reveals that in Euclidean Y^n, the connection is determined entirely by the torsion tensor, illustrating how torsion can introduce non-trivial parallel transport even in a flat metric space.

2.3. *Covariant Derivatives*

For a vector u^i, the covariant derivative is:

$$u^i_{;k} = \partial_k u^i + \Gamma^i_{jk} u^j. \tag{2.11}$$

The commutator of covariant derivatives reveals the torsion and curvature:

$$u^i_{;j;k} - u^i_{;k;j} = R^i_{ljk} u^l + S^l_{jk} u^i_{;l}, \tag{2.12}$$

where R^i_{ljk} is the curvature tensor.

3. Parallel Transport and Geometric Meaning of Torsion

3.1. *Closure Failure of Infinitesimal Parallelograms*

A fundamental geometric interpretation of torsion emerges when considering parallel transport around infinitesimal loops. In Euclidean space Y^n with torsion, consider an infinitesimal parallelogram defined by two vectors $A^i\tau$ and $B^j\tau$, where τ is an infinitesimal parameter.

The conventional process of parallel transport in Riemannian geometry (where torsion vanishes) preserves the closure of such parallelograms. However, in the presence of torsion, when we parallel transport the vector $A^i\tau$ along $B^j\tau$ and then parallel transport $B^j\tau$ along $A^i\tau$, the parallelogram fails to close. As shown by [20], the resulting gap vector Z^k, measured to second order in τ, is:

$$Z^k = S^k_{ij} A^i B^j \tau^2. \tag{3.1}$$

The squared magnitude of this gap in the Euclidean metric ($g_{ij} = \delta_{ij}$) is:

$$|Z|^2 = g_{pq} Z^p Z^q = g_{pq} S^p_{ij} S^q_{kl} A^i B^j A^k B^l \tau^4. \tag{3.2}$$

For the Euclidean case with $g_{ij} = \delta_{ij}$, this simplifies to:

$$|Z|^2 = \delta_{pq} S^p_{ij} S^q_{kl} A^i B^j A^k B^l \tau^4. \tag{3.3}$$

Equation (3.1) provides an operational definition of the torsion tensor: S^k_{ij} measures the non-closure of infinitesimal parallelograms under parallel transport. This geometric interpretation distinguishes torsion from curvature; while curvature measures the rotation of a vector when transported around a closed loop, torsion measures the translation or displacement resulting from such transport.

The gap vector Z^k has several important properties:

(1) It is bilinear in the vectors A^i and B^j.
(2) It is antisymmetric: reversing the order of transport (i.e., swapping A^i and B^j) changes the sign of Z^k, consistent with the antisymmetry of S^k_{ij}.
(3) It vanishes if and only if torsion is zero, recovering the closure property of Riemannian geometry.

This geometric manifestation of torsion offers an intuitive understanding of its physical significance in unified field theories, where it may be associated with intrinsic spin or other microscopic properties of matter.

3.2. *Parallel Transport Equations*

In the space Y^n with Euclidean metric and torsion, the parallel transport of a vector V^i along a curve parameterized by τ with tangent vector $u^k = dx^k/d\tau$ is governed by the condition:

$$\frac{DV^i}{d\tau} = \frac{dV^i}{d\tau} + \Gamma^i_{jk} V^j u^k = 0, \tag{3.3}$$

where Γ^i_{jk} is the connection given by (2.9). This equation ensures that the vector V^i is transported along the curve without change with respect to the connection.

In standard Euclidean geometry without torsion ($\Gamma^i_{jk} = 0$ in Cartesian coordinates), parallel transport preserves the vector's components, making it path-independent. However, in Y^n with torsion, even though the metric is flat ($g_{ij} = \delta_{ij}$), the connection coefficients Γ^i_{jk} do not vanish in general. Substituting the expression (2.9) for the connection yields:

$$\frac{DV^i}{d\tau} = \frac{dV^i}{d\tau} + \frac{1}{2}\left(S^i_{jk} + S^j_{ki} + S^k_{ji}\right) V^j u^k = 0. \tag{3.4}$$

Thus, the presence of torsion introduces a coupling between the transported vector and the tangent direction of the curve, causing the vector to rotate as it is transported.

The non-zero connection coefficients imply that parallel transport becomes path-dependent. Specifically, if a vector is transported along two different curves connecting the same two points, the resulting vectors at the endpoint may differ. This path-dependence is directly related to the closure failure discussed in Section 3.1: transporting a vector around an infinitesimal parallelogram does not return it to its original value, with the discrepancy given by (3.1).

To see this connection explicitly, consider an infinitesimal closed loop. The change in a vector V^i when transported around such a loop can be expressed in terms of the curvature and torsion tensors. In Y^n with Euclidean metric, the curvature tensor is generally non-zero due to torsion, contributing to the path-dependence. However, even in the absence of curvature (if the torsion is constant, for example), the torsion itself leads to a translation of the vector, as captured by the gap vector Z^k in (3.1).

Thus, the parallel transport equation (3.3) highlights a fundamental departure from Riemannian geometry: in Y^n with torsion, the notion of "keeping a vector constant" along a curve depends on the path, reflecting the richer geometric structure induced by torsion.

4. Geodesics in Euclidean Space with Torsion

4.1. *Geodesic Equation*

In the space Y^n with Euclidean metric and torsion, geodesics are defined as curves whose tangent vector is parallel transported along the curve itself. Taking the tangent vector as $V^i = dx^i/d\tau$ in the parallel transport equation (3.3), we obtain the geodesic equation:

$$\frac{d^2x^k}{d\tau^2} = -\Gamma^k_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}. \tag{4.1}$$

This is the standard form of the geodesic equation, but with the crucial difference that the connection Γ^k_{ij} now includes torsion contributions. Substituting the explicit expression for the connection in Euclidean Y^n from (2.9) yields:

$$\frac{d^2x^k}{d\tau^2} = -\frac{1}{2}\left(S^k_{ij} + S^i_{jk} + S^j_{ik}\right)\dot{x}^i\dot{x}^j, \tag{4.2}$$

where we denote $\dot{x}^i = dx^i/d\tau$. Noting that S^k_{ij} is antisymmetric in i and j, we have $S^k_{ij}\dot{x}^i\dot{x}^j = 0$, since the product $\dot{x}^i\dot{x}^j$ is symmetric. Therefore, (4.2) simplifies to:

$$\frac{d^2x^k}{d\tau^2} = -\left(S^i_{jk} + S^j_{ik}\right)\dot{x}^i\dot{x}^j. \tag{4.3}$$

This equation reveals that in Euclidean Y^n with torsion, geodesics are generally not straight lines (for which $d^2x^k/d\tau^2 = 0$). The torsion tensor acts as a force field that deflects particles from inertial motion, even in the absence of metric curvature.

Theorem 4.1. *In Euclidean Y^n with torsion, geodesics are straight lines if and only* *if the torsion tensor satisfies the condition:*

$$S^i_{jk} + S^j_{ik} = 0 \quad \textit{for all } i, j, k.$$

This condition is equivalent to the total antisymmetry of the torsion tensor, i.e., $S_{ijk} = S_{[ijk]}$.

Proof. If S_{ijk} is totally antisymmetric, then $S^i_{jk} = \delta^{il} S_{ljk}$ is antisymmetric in all indices. In particular, it is antisymmetric in i and j, so $S^i_{jk}\dot{x}^i\dot{x}^j = 0$ for any $\dot{x}^i$. The same holds for S^j_{ik}. Hence, the right-hand side of (4.3) vanishes, and the geodesic equation reduces to $d^2x^k/d\tau^2 = 0$, which describes straight lines.

Conversely, if geodesics are straight lines for all possible initial velocities, then (4.3) must vanish identically. This implies that the tensor $T^i_{jk} = S^i_{jk} + S^j_{ik}$ must be zero. Contracting T^i_{jk} with arbitrary vectors and using the antisymmetry of S^i_{jk} in its lower indices leads to the conclusion that S_{ijk} must be totally antisymmetric. □

The theorem highlights an important special case: when torsion is totally antisymmetric, it does not affect the geodesic motion of spinless particles. This result is consistent with the Einstein-Cartan theory, where totally antisymmetric torsion couples only to the spin of particles, not to their mass.

For general torsion, however, equation (4.3) shows that geodesics deviate from straight lines. This deviation is quadratic in the velocity, unlike the linear velocity dependence of the Lorentz force in electromagnetism. This suggests that if torsion is to represent electromagnetic effects, additional assumptions or modifications to the geodesic equation may be necessary, such as introducing a coupling constant or considering the motion of charged particles in a different manner.

4.2. *Variational Principle for Geodesics*

In Riemannian geometry, geodesics can be derived from a variational principle as curves that extremize the arc length between two points. In the space Y^n with Euclidean metric and torsion, the arc length of a non-isotropic curve $x^i(t)$ (where the tangent vector is never null) is given by:

$$s = \int_{t_1}^{t_2} \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\, dt. \tag{4.4}$$

For the Euclidean metric $g_{ij} = \delta_{ij}$, this reduces to the standard expression for arc length in $\mathbb{R}^n$.

In the absence of torsion, the condition $\delta s = 0$ yields the geodesic equation, which in Euclidean space corresponds to straight lines. However, in the presence of torsion, the variation of the arc length acquires an additional term. Following [20], the first variation of s is:

$$\delta s = \int_{t_1}^{t_2} g_{ij} S^j_{pk} \frac{dx^i}{dt}\frac{dx^p}{dt}\delta x^k\, dt. \tag{4.5}$$

This result can be derived by considering the standard variation of the arc length functional and taking into account the torsion-dependent part of the connection. The presence of the torsion tensor S^j_{pk} in the integrand indicates that the extremization of arc length is affected by torsion.

The variational principle $\delta s = 0$ for arbitrary variations δx^k leads to the condition:

$$g_{ij} S^j_{pk} \frac{dx^i}{dt} \frac{dx^p}{dt} = 0. \tag{4.6}$$

For a non-degenerate metric, this implies that $S^j_{pk} \dot{x}^i \dot{x}^p = 0$ for all i, k. This condition is generally not satisfied unless the torsion tensor vanishes or has specific algebraic properties.

Therefore, in Y^n with torsion, curves that extremize arc length (i.e., satisfy $\delta s = 0$) are not necessarily geodesics as defined by the parallel transport of their tangent vectors. This is a departure from Riemannian geometry, where the two definitions coincide. Only when the torsion tensor satisfies (4.6) for all possible tangent vectors do the extremal curves coincide with geodesics. In particular, if $S^j_{pk} = 0$, we recover the classical result.

This discrepancy highlights the richer geometric structure of spaces with torsion. The variational principle based on arc length does not, in general, yield the same curves as the geodesic equation (4.3). This suggests that in such spaces, the principle of extremal length might not correspond to the motion of free particles, which are typically described by the geodesic equation. Alternatively, one might consider a different action functional that leads to the geodesic equation in the presence of torsion, such as one involving the connection directly.

In summary, the variational principle for arc length in Euclidean Y^n with torsion yields an additional torsion-dependent term. As a result, the curves that extremize length are not generally geodesics, indicating a fundamental difference between the geometric and variational definitions of "straight lines" in spaces with torsion.

5. Curvature Generated by Torsion

5.1. *Curvature Tensor*

In differential geometry, the curvature tensor measures the failure of parallel transport to commute when moving around infinitesimal loops. In the space Y^n with Euclidean metric and torsion, the curvature tensor is defined in the standard way:

$$R^p_{ikl} = \partial_k \Gamma^p_{li} - \partial_l \Gamma^p_{ki} + \Gamma^p_{kq} \Gamma^q_{li} - \Gamma^p_{lq} \Gamma^q_{ki}. \tag{5.1}$$

Although the metric is flat ($g_{ij} = \delta_{ij}$), the curvature tensor can be non-zero because the connection Γ^p_{kl} depends on the torsion tensor, which may vary from point to point. This is a key distinction from Riemannian geometry, where a flat metric implies zero curvature. In Y^n, the torsion introduces an additional source of curvature, even in the absence of metric curvature.

Substituting the expression for the connection in Euclidean Y^n from (2.9) into (5.1) yields the curvature tensor in terms of the torsion tensor and its derivatives. After straightforward but lengthy algebra, we obtain:

$$\begin{aligned} R^p_{ikl} = \frac{1}{4}\Big[& \partial_k \left(S^p_{li} + S^l_{ip} + S^i_{pl} \right) - \partial_l \left(S^p_{ki} + S^k_{ip} + S^i_{pk} \right) \\ & + \left(S^p_{kq} + S^k_{qp} + S^q_{pk} \right) \left(S^q_{li} + S^l_{iq} + S^i_{ql} \right) \\ & - \left(S^p_{lq} + S^l_{qp} + S^q_{pl} \right) \left(S^q_{ki} + S^k_{iq} + S^i_{qk} \right) \Big]. \end{aligned} \tag{5.2}$$

This expression reveals that the curvature tensor in Y^n depends on both the torsion tensor and its first derivatives. Even if the torsion is constant, the quadratic terms in torsion can still produce non-zero curvature.

From the curvature tensor, we can form the Ricci tensor by contraction:

$$R_{ik} = R^p_{ipk}. \tag{5.3}$$

In Euclidean Y^n, this contraction simplifies due to the flat metric, but the resulting Ricci tensor is generally non-zero if torsion is present.

The curvature tensor in a space with torsion also satisfies a generalized Bianchi identity. However, unlike in Riemannian geometry, this identity includes torsion terms. In [20], author provides a detailed discussion of these identities in the general context of Y^n.

The fact that curvature can be non-zero in a space with a flat metric but non-zero torsion has important implications. It means that geometric effects traditionally associated with curvature, such as geodesic deviation, can arise purely from torsion. This challenges the Riemannian paradigm where curvature is solely determined by the metric. In Y^n, both metric and torsion contribute to the curvature, offering a richer geometric framework for unified field theories.

In the next subsection, we will examine a concrete example of constant torsion in three dimensions, which illustrates how torsion generates constant curvature in an otherwise flat space.

5.2. *Example: Constant Totally Antisymmetric Torsion in 3D*

To illustrate how torsion can generate curvature in an otherwise flat space, consider the specific case of three-dimensional Euclidean Y^3 with constant totally antisymmetric torsion. In Cartesian coordinates, let the torsion tensor be given by:

$$S_{ijk} = T\epsilon_{ijk}, \quad T \in \mathbb{R}, \tag{5.3}$$

where ϵ_{ijk} is the Levi-Civita symbol (completely antisymmetric with $\epsilon_{123} = 1$), and T is a constant. This form of torsion is particularly significant in physics, as totally antisymmetric torsion appears naturally in string theory and supergravity.

Substituting (5.3) into the expression for the connection in Euclidean Y^n (2.9), and using the properties of the Levi-Civita symbol, we obtain:

$$\Gamma^p_{kl} = \frac{3T}{2}\epsilon^p{}_{kl}, \tag{5.4}$$

where indices are raised with the Euclidean metric δ^{ij}. Note that $\epsilon^{p}{}_{kl} = \delta^{pm}\epsilon_{mkl}$.

To compute the curvature tensor, we substitute (5.4) into the definition (5.1). Since the torsion is constant, the derivative terms in (5.2) vanish, leaving only the quadratic terms. After simplification, we find:

$$R^{p}_{ikl} = \frac{9T^2}{4}\left(\epsilon^{p}{}_{kq}\epsilon^{q}{}_{li} - \epsilon^{p}{}_{lq}\epsilon^{q}{}_{ki}\right). \tag{5.5}$$

Using the well-known identity for the product of two Levi-Civita symbols in three dimensions:

$$\epsilon^{p}{}_{kq}\epsilon^{q}{}_{li} = \delta^{p}_{l}\delta_{ki} - \delta^{p}_{i}\delta_{kl}, \tag{5.6}$$

the curvature tensor simplifies to:

$$R^{p}_{ikl} = \frac{9T^2}{4}\left(\delta^{p}_{l}\delta_{ki} - 2\delta^{p}_{i}\delta_{kl} + \delta^{p}_{k}\delta_{il}\right). \tag{5.7}$$

Contracting the first and third indices to form the Ricci tensor, we obtain:

$$R_{ik} = R^{p}_{ipk} = \frac{9T^2}{2}\delta_{ik}. \tag{5.8}$$

This result demonstrates a remarkable phenomenon: **constant totally antisymmetric torsion induces constant positive Ricci curvature in an otherwise Euclidean space**. The Ricci curvature is proportional to T^2 and has the form of a cosmological constant term in Einstein's equations.

Several important observations follow from this example:

(1) The curvature is non-zero even though the metric is flat. This contradicts the Riemannian intuition that a flat metric implies zero curvature.
(2) The curvature is constant and isotropic, reflecting the homogeneity and isotropy of the constant torsion field.
(3) The sign of the curvature is positive, regardless of the sign of T, since it depends on T^2.
(4) This example provides a concrete realization of how torsion can mimic the effects of metric curvature, a key idea in teleparallel formulations of gravity.

The scalar curvature $R = g^{ik}R_{ik}$ in this case is:

$$R = \delta^{ik}\left(\frac{9T^2}{2}\delta_{ik}\right) = \frac{27T^2}{2}. \tag{5.9}$$

Thus, the entire space has constant positive scalar curvature proportional to T^2.

This example clearly illustrates one of the central themes of this paper: in spaces with torsion, the geometric structure is richer than in Riemannian spaces. Curvature can arise from torsion alone, opening up new possibilities for geometric descriptions of physical fields. In particular, it suggests that what we normally attribute to metric curvature in general relativity might, in alternative formulations, be encoded in torsion.

5.3. *Ricci-Jacobi Identity*

In Riemannian geometry, the Riemann curvature tensor satisfies the first Bianchi identity, also known as the Ricci identity, which in the absence of torsion takes the form $R^p_{[ikl]} = 0$. However, in spaces with torsion, this identity is modified to include torsion terms. For the space Y^n with Euclidean metric and torsion, the Riemann tensor satisfies a generalized Ricci-Jacobi identity, as derived by [20]:

$$R^p_{ikl} + R^p_{kil} + R^p_{lik} = S^p_{ik;l} + S^p_{kl;i} + S^p_{li;k} + S^p_{iq}S^q_{kl} + S^p_{kq}S^q_{li} + S^p_{lq}S^q_{ik}. \tag{5.10}$$

Here, the semicolon denotes the covariant derivative with respect to the connection Γ^p_{kl}, which includes torsion. The left-hand side is the cyclic sum over the last three indices of the Riemann tensor, while the right-hand side consists of cyclic sums of the covariant derivative of the torsion tensor and quadratic terms in the torsion tensor.

This identity reduces to the standard first Bianchi identity of Riemannian geometry when torsion vanishes. It is an important algebraic relation that constrains the curvature and torsion tensors, and it plays a crucial role in the derivation of field equations from a variational principle.

In the case of constant torsion, the covariant derivatives of the torsion tensor vanish (since the partial derivatives are zero and the connection terms cancel out due to the antisymmetry of torsion). Then the identity simplifies to:

$$R^p_{ikl} + R^p_{kil} + R^p_{lik} = S^p_{iq}S^q_{kl} + S^p_{kq}S^q_{li} + S^p_{lq}S^q_{ik}. \tag{5.11}$$

This simplified identity can be verified for the example of constant totally antisymmetric torsion in 3D. Substituting the torsion tensor $S_{ijk} = T\epsilon_{ijk}$ and the curvature tensor (5.7) into (5.11) confirms the validity of the identity in that specific case.

The Ricci-Jacobi identity (5.10) is one of several Bianchi-type identities that hold in spaces with torsion. The other identities include the second Bianchi identity, which also acquires torsion terms, and the Bianchi identity for the torsion tensor itself. These identities are essential for understanding the integrability conditions and the conservation laws in theories with torsion.

In summary, the Ricci-Jacobi identity illustrates how the presence of torsion modifies the algebraic properties of the curvature tensor. This modification has profound implications for the geometry of Y^n and for the formulation of physical theories in such spaces.

6. Field Equations from Variational Principle

6.1. *Action Functional*

The field equations for the space Y^n with Euclidean metric and torsion can be derived from a variational principle. Following [20], we consider an action functional that depends on both the metric and the torsion tensor. A natural choice, which reduces to the Einstein–Hilbert action when torsion vanishes, is:

$$\mathcal{S} = \int \left(R_{ik} + S^j_{il} S^l_{kj} \right) g^{ik} \sqrt{-g}\, d^n x, \tag{6.1}$$

where $R_{ik} = R^p_{ipk}$ is the Ricci tensor formed from the curvature tensor (5.1), and S^j_{il} is the torsion tensor. The integrand consists of two parts: the Ricci scalar term $R = g^{ik} R_{ik}$, which is the usual curvature scalar, and a quadratic torsion term $S^j_{il} S^l_{kj} g^{ik}$. The latter can be viewed as a "kinetic" term for the torsion field, analogous to the $F_{\mu\nu} F^{\mu\nu}$ term in electromagnetism.

In the Euclidean setting with metric $g_{ij} = \delta_{ij}$, the determinant $g = \det(g_{ij}) = 1$, so $\sqrt{-g} = 1$ (the minus sign is irrelevant in the Euclidean signature). Moreover, the inverse metric is simply $g^{ik} = \delta^{ik}$. Consequently, the action (6.1) simplifies to:

$$\mathcal{S} = \int \left(R_{ik} + S^j_{il} S^l_{kj} \right) \delta^{ik}\, d^n x. \tag{6.2}$$

The variation of this action with respect to the independent fields—the metric and the torsion (or equivalently the connection)—yields the field equations that govern the geometry of Y^n. It is important to note that, in the presence of torsion, the connection is not uniquely determined by the metric; therefore, we treat the metric and the connection (or the torsion) as independent variables in the variational principle.

The action (6.1) is a special case of more general metric–affine actions studied in the literature. Its choice is motivated by the desire to obtain field equations that are second order in derivatives and that generalize the Einstein equations to include torsion. In the following subsections, we will perform the variations and derive the corresponding field equations.

6.2. *Variation with Respect to Metric and Connection*

The action functional (6.1) depends on two independent geometric objects: the metric tensor g_{ij} and the connection Γ^k_{ij} (or equivalently, the torsion tensor S^k_{ij}). To derive the field equations, we perform independent variations with respect to these variables.

6.2.1. Variation with Respect to the Metric

Varying the action with respect to the inverse metric g^{ik} while keeping the connection fixed yields:

$$\delta_g \mathcal{S} = \int \left(R_{ik} + S^j_{il} S^l_{kj} \right) \sqrt{-g}\, \delta g^{ik}\, d^n x = 0, \tag{6.3}$$

where we have used the fact that $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}\, g_{ik} \delta g^{ik}$ and that the variation of the integrand with respect to g^{ik} gives the expression in parentheses. Since δg^{ik} is arbitrary, we obtain the field equation:

$$R_{ik} + S^j_{il} S^l_{kj} = 0. \tag{6.4}$$

In the Euclidean case with $g_{ij} = \delta_{ij}$, this equation simplifies to:

$$R_{ik} + S^j_{il} S^l_{kj} = 0, \tag{6.5}$$

where the indices are raised and lowered with the Kronecker delta.

For the example of constant totally antisymmetric torsion in 3D, substituting (5.8) and (5.3) into (6.5) gives:

$$\frac{9T^2}{2}\delta_{ik} + T^2 \epsilon_{jil}\epsilon_{lkj} = 0.$$

Using the identity $\epsilon_{jil}\epsilon_{lkj} = -2\delta_{ik}$, we obtain:

$$\left(\frac{9T^2}{2} - 2T^2\right)\delta_{ik} = \frac{5T^2}{2}\delta_{ik} = 0,$$

which forces $T = 0$. Thus, constant totally antisymmetric torsion cannot satisfy the vacuum field equation (6.5). This indicates that in a vacuum, torsion must be dynamical rather than constant.

6.2.2. Variation with Respect to the Connection

Varying the action with respect to the connection Γ^k_{ij} (or equivalently, the torsion tensor S^k_{ij}) is more involved. After a lengthy calculation [20], one obtains the following field equation:

$$\begin{aligned} g^{ij}_{,k} - \frac{1}{2} g^{ij} g_{pq} g^{pq}_{,k} + g^{pj}\Gamma^i_{pk} + g^{ip}\Gamma^j_{kp} & \\ - \delta^i_k \left(g^{ip}_{,p} - \frac{1}{2} g^{ip} g_{mn} g^{mn}_{,p} + g^{pq}\Gamma^i_{pq} \right) & \\ - g^{ij}\Gamma^p_{kp} + 2\left(g^{ip} S^j_{pk} + g^{pj} S^i_{kp} \right) &= 0. \end{aligned} \tag{6.6}$$

This equation can be simplified by contracting indices. Contracting i and k yields an identity, while contracting j and k gives:

$$-3g^{ij}_{,j} + \frac{3}{2} g^{ij} g_{pq} g^{pq}_{,j} - 3g^{pq}\Gamma^i_{pq} + g^{ip} S^l_{pj} = 0. \tag{6.7}$$

Using (6.7), equation (6.6) can be rewritten in a more symmetric form. For the Euclidean metric $g_{ij} = \delta_{ij}$, the partial derivatives of the metric vanish, and (6.6) reduces to:

$$\Gamma^i_{jk} + \Gamma^j_{ik} - \delta^i_k \Gamma^p_{jp} - \delta^j_k \Gamma^p_{ip} + 2\left(S^i_{jk} + S^j_{ik}\right) = 0. \tag{6.8}$$

Substituting the expression for the connection (2.9) into (6.8) yields constraints on the torsion tensor. These constraints, together with equation (6.5), form the complete set of field equations for Euclidean Y^n with torsion.

In summary, the variational principle leads to two sets of field equations: one from the metric variation, which couples curvature and torsion quadratically, and one from the connection variation, which imposes compatibility conditions between the metric and torsion. In the following subsection, we will explore the physical interpretation of these equations.

6.2.3. Metric Variation

The variation of the action (6.1) with respect to the inverse metric g^{ik} yields the field equation:

$$R_{ik} + S^j_{il} S^l_{kj} = 0. \tag{6.3}$$

This equation couples the Ricci tensor, which encodes curvature, to a quadratic combination of the torsion tensor. In the Euclidean case with $g_{ij} = \delta_{ij}$, the Ricci tensor is given by (5.8) for constant totally antisymmetric torsion.

For the 3D constant torsion example discussed in Section 5.2, we substitute the Ricci tensor (5.8) and compute the quadratic torsion term:

$$S^j_{il} S^l_{kj} = T^2 \epsilon_{jil} \epsilon_{lkj} = -2T^2 \delta_{ik}, \tag{6.4}$$

where we have used the identity $\epsilon_{jil}\epsilon_{lkj} = -2\delta_{ik}$. Substituting these into (6.3) gives:

$$\left(\frac{9T^2}{2} - 2T^2\right)\delta_{ik} = \frac{5T^2}{2}\delta_{ik} = 0 \quad \Rightarrow \quad T = 0. \tag{6.5}$$

This result shows that constant totally antisymmetric torsion cannot satisfy the vacuum field equation (6.3) in three-dimensional Euclidean Y^3. Therefore, if torsion is present in vacuum, it must be dynamical—varying from point to point—rather than constant. This is analogous to the situation in electromagnetism, where a constant electromagnetic field does not satisfy Maxwell's equations in vacuum without sources.

The condition $T = 0$ in this example suggests that the simple action (6.1) does not admit constant torsion solutions in vacuum. This may indicate the need for additional terms in the action or the inclusion of matter sources to support a constant torsion field.

6.2.4. Connection Variation

Varying the action with respect to the connection Γ^k_{ij} yields the compatibility condition between the metric and torsion. Following the derivation of [20], we obtain:

$$g_{ij;k} - \frac{1}{2} g_{ij} g^{pq} g_{pq;k} + g_{ip} \Gamma^p_{kj} + g_{jp} \Gamma^p_{ik} - g_{ij} \Gamma^p_{kp} - \frac{1}{3} g_{jk} S^p_{ip} + 2 \left(g_{pj} S^p_{ik} + g_{ip} S^p_{kj} \right) = 0. \quad (6.6)$$

Here, the semicolon denotes the covariant derivative with respect to the connection Γ^p_{kl}. This equation ensures that the connection is compatible with both the metric and the torsion in the sense that it extremizes the action.

For the Euclidean metric $g_{ij} = \delta_{ij}$, the covariant derivatives of the metric vanish, and (6.6) simplifies considerably. Using $\Gamma_{ijk} = \delta_{kl} \Gamma^l_{ij}$, we obtain:

$$\Gamma_{ijk} + \Gamma_{jik} - \delta_{ij} \Gamma^p_{kp} - \frac{1}{3} \delta_{jk} S^p_{ip} + 2 \left(S_{kij} + S_{ikj} \right) = 0. \quad (6.7)$$

By contracting indices in (6.7), we can derive constraints on the torsion tensor. For instance, contracting i and j yields:

$$2\Gamma^i_{ik} - n\Gamma^p_{kp} - \frac{1}{3} S^p_{kp} + 2 \left(S_{kii} + S_{iki} \right) = 0, \quad (1)$$

which further simplifies using the properties of the torsion tensor. These constraints must be satisfied by any torsion field in Euclidean Y^n.

Equation (6.6) (or its Euclidean counterpart (6.7)) complements the field equation obtained from the metric variation. Together, they determine the geometry of Y^n by relating the metric, connection, and torsion.

6.3. *Generalized Einstein-Hilbert Equations with Matter*

To incorporate matter fields into the theory, we add a matter action $\mathcal{S}_m$ to the geometric action (6.1). The total action becomes $\mathcal{S}_{\text{total}} = \mathcal{S} + \mathcal{S}_m$, where $\mathcal{S}_m$ depends on the matter fields and possibly on the metric and torsion. The energy-momentum tensor of matter is defined as:

$$T_{ik} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}_m}{\delta g^{ik}}.$$

Varying the total action with respect to the inverse metric g^{ik} yields the field equations in the presence of matter. Following citeYaremenko2014, the resulting equations are:

$$\begin{aligned} R_{ik} - \frac{1}{2} g_{ik} R + S^p_{ip} S^q_{qk} + S^q_{ip} S^p_{kq} + g_{kp} g^{qt} S^m_{it} S^p_{mq} \\ + S^j_{il} S^l_{kj} - \frac{1}{2} S^j_{pl} S^l_{kq} g^{pq} g_{ik} = K T_{ik}, \end{aligned} \quad (6.8)$$

where K is a coupling constant that depends on the gravitational constant and the details of the matter coupling. The left-hand side of (6.8) generalizes the Einstein tensor $G_{ik} = R_{ik} - \frac{1}{2}g_{ik}R$ by adding terms quadratic in the torsion tensor. These additional terms represent the contribution of torsion to the effective stress-energy.

In the vacuum case where $T_{ik} = 0$, equation (6.8) reduces to:

$$R_{ik} - \frac{1}{2}g_{ik}R + S^{p}_{ip}S^{q}_{qk} + S^{q}_{ip}S^{p}_{kq} + g_{kp}g^{qt}S^{m}_{it}S^{p}_{mq} + S^{j}_{il}S^{l}_{kj} - \frac{1}{2}S^{j}_{pl}S^{l}_{kq}g^{pq}g_{ik} = 0. \quad (6.9)$$

This vacuum equation is more complex than the simple equation (6.3) obtained from the metric variation of the pure geometric action. The difference arises because in the full variational principle, the connection (or torsion) is also varied independently, and the resulting connection field equation (6.6) must be satisfied simultaneously. Combining (6.6) and (6.9) can lead to simplifications. In particular, taking the trace of (6.9) and using (6.6) may reproduce (6.3) under certain conditions. However, a detailed analysis of these equations in the Euclidean case is beyond the scope of this paper.

Equation (6.8) represents the analogue of the Einstein field equations in the space Y^n with torsion. It suggests that both curvature and torsion contribute to the geometry and that matter acts as a source for this combined geometric structure. This framework provides a unified description of gravitational and possibly other interactions, such as electromagnetism, if the torsion tensor is suitably interpreted.

7. Physical Interpretation and Applications

7.1. *Torsion as a Force Field*

The geodesic equation in Euclidean Y^n with torsion, given by (4.3), can be rewritten as:

$$\ddot{x}^k = -\left(S^{i}_{jk} + S^{j}_{ik}\right)\dot{x}^i\dot{x}^j, \quad (2)$$

where dots denote derivatives with respect to the parameter τ. This equation describes the acceleration of a particle moving along a geodesic. Unlike in classical mechanics where forces are typically linear in velocity, the right-hand side is quadratic in the velocity $\dot{x}^i$. This indicates that torsion acts as a velocity-dependent force.

A natural comparison can be made with the Lorentz force law in electromagnetism, which describes the motion of a charged particle in an electromagnetic field:

$$\ddot{x}^k = \frac{q}{m}F^{k}_{\ i}\dot{x}^i, \quad (3)$$

where q is the charge, m the mass, and F_{ij} the electromagnetic field strength tensor. The Lorentz force is linear in the velocity. In contrast, the torsion-induced force is quadratic.

To relate the two, one might tentatively identify:

$$F_{ij} \sim S_{ijk}\dot{x}^k, \quad (7.1)$$

but this identification is problematic because the right-hand side depends on the velocity, whereas F_{ij} in electromagnetism is a field independent of the particle's state of motion. Moreover, the force in (7.1) would be antisymmetric in i, j (since S_{ijk} is antisymmetric in the first two indices) and linear in velocity, which is closer to the Lorentz force. However, our geodesic equation (4.3) is quadratic in velocity and involves a different combination of torsion components.

A more consistent geometric approach to unifying electromagnetism with gravity using torsion might involve introducing a coupling between the torsion tensor and the electromagnetic field strength. For example, in Einstein–Cartan theory, torsion is coupled to the spin density of matter, not directly to the electromagnetic field. Alternatively, in some Kaluza–Klein theories, torsion components in extra dimensions can be related to electromagnetic potentials.

Despite the differences, the analogy suggests that torsion could represent a generalized force field. The quadratic dependence on velocity might indicate that torsion affects particles differently depending on their energy or momentum, which could have implications for high-energy physics or cosmology.

In summary, while the geodesic equation in Y^n with torsion shows that torsion can exert a force on particles, the exact relationship with known force fields like electromagnetism requires additional structure and assumptions. This opens up interesting avenues for further research on the physical interpretation of torsion.

7.2. *Electromagnetic Interpretation*

A longstanding aspiration in theoretical physics has been the geometric unification of gravity and electromagnetism. In the context of Euclidean Y^n with torsion, one possible approach is to identify the torsion tensor with the electromagnetic field strength. Consider the ansatz:

$$S_{ijk} = F_{ij} u_k, \tag{7.2}$$

where F_{ij} is the antisymmetric electromagnetic field strength tensor and u_k is a fixed vector field, often interpreted as a cosmic vector or the velocity field of a background medium. This ansatz imposes a particular algebraic structure on the torsion tensor, constraining it to be "simple" in the sense that it factorizes into an antisymmetric tensor and a vector.

Substituting (7.2) into the vacuum field equation obtained from the metric variation, equation (6.3), we compute:

$$S^j_{il} S^l_{kj} = (F^j{}_i u_l)(F^l{}_k u_j) = F_{ji} u_l F^l{}_k u^j,$$

where indices are raised and lowered with the Euclidean metric δ_{ij}. After rearranging, we obtain:

$$S^j_{il} S^l_{kj} = (F_{ji} u^j)(F_{lk} u^l) = E_i E_k,$$

where we define $E_i = F_{ji} u^j$, which can be interpreted as an electric field component relative to the vector u_k. The vacuum field equation (6.3) then becomes:

$$R_{ik} + E_i E_k = 0. \tag{7.3}$$

This equation shows that the Ricci curvature is directly sourced by the square of the electric field. In the weak-field limit, where the curvature is small and the electromagnetic field is weak, equation (7.3) may approximate the coupled Einstein–Maxwell equations in a specific gauge or under particular symmetry conditions.

However, it is important to note several limitations of this interpretation:

1. The ansatz (7.2) restricts the torsion tensor to a specific form, which may not be general enough to describe arbitrary electromagnetic configurations. 2. The resulting equation (7.3) lacks the full structure of the electromagnetic energy-momentum tensor, which typically includes both electric and magnetic contributions and a trace term. 3. The fixed vector u_k introduces a preferred direction, breaking the isotropy of space unless it is a Killing vector of the background.

Despite these limitations, the electromagnetic interpretation of torsion offers an intriguing geometric perspective. It suggests that the electromagnetic field may be understood as part of the spacetime geometry, specifically as the torsion of the connection. This idea resonates with earlier unified field theories proposed by Einstein, Weyl, and others.

In conclusion, while the identification (7.2) leads to an equation that resembles a simplified version of the Einstein–Maxwell system, a complete and consistent geometric unification of gravity and electromagnetism within the framework of Euclidean Y^n with torsion would require a more elaborate construction, possibly involving additional fields or modified action principles.

8. Conclusions

We have systematically analyzed Euclidean space endowed with torsion, within the framework of metric-affine geometry. Key findings include:

(1) The connection in Cartesian coordinates depends solely on the torsion tensor.
(2) Geodesics deviate from straight lines unless torsion is totally antisymmetric (Theorem 4.1).
(3) Torsion generates curvature even when the metric is flat, as demonstrated by the constant curvature induced by constant totally antisymmetric torsion in 3D.
(4) Field equations derived from a variational principle constrain torsion dynamically, forbidding constant vacuum solutions and suggesting wave-like propagation.

This simplified model serves as a pedagogical bridge to more advanced gauge theories of gravity and unification. Future research directions include:

- Coupling torsion to matter fields (spinors, currents).
- Studying wave solutions for torsion in Euclidean space.
- Exploring embeddings in higher-dimensional spaces.
- Investigating the quantum aspects of torsion fields.

References

[1] Aldrovandi, R. and Pereira, J.G. (2013), *Teleparallel Gravity: An Introduction*, Springer.

[2] Blagojević, M. and Hehl, F.W. (Eds.) (2013), *Gauge Theories of Gravitation: A Reader with Commentaries*, Imperial College Press.

[3] Capozziello, S. and Lambiase, G. (2000), "Electromagnetic and gravitational interactions in the Einstein–Cartan theory", *General Relativity and Gravitation*, Vol. 32, pp. 673–696.

[4] Cartan, É. (1922), "Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion", *Comptes Rendus de l'Académie des Sciences Paris*, Vol. 174, pp. 593–595.

[5] de Andrade, V.C. and Pereira, J.G. (1999), "Torsion and the electromagnetic field", *International Journal of Modern Physics D*, Vol. 8, pp. 141–151.

[6] Einstein, A. (1920), *Relativity: The Special and General Theory*, Henry Holt and Company, New York.

[7] Einstein, A. (1921), *The Meaning of Relativity*, Princeton University Press, Princeton.

[8] Einstein, A. (1930), "Théorie unitaire de champ physique", *Annales de l'Institut Henri Poincaré*, Vol. 1, pp. 1–24.

[9] Hammond, R.T. (2002), "Torsion gravity", *Reports on Progress in Physics*, Vol. 65, pp. 599–649.

[10] Hayashi, K. and Shirafuji, T. (1979), "New general relativity", *Physical Review D*, Vol. 19, pp. 3524–3553.

[11] Hehl, F.W., von der Heyde, P., Kerlick, G.D. and Nester, J.M. (1976), "General relativity with spin and torsion: Foundations and prospects", *Reviews of Modern Physics*, Vol. 48, pp. 393–416.

[12] Jiménez, J.B., Heisenberg, L. and Koivisto, T.S. (2019), "The geometrical trinity of gravity", *Universe*, Vol. 5, No. 7, Article 173.

[13] Kibble, T.W.B. (1961), "Lorentz invariance and the gravitational field", *Journal of Mathematical Physics*, Vol. 2, pp. 212–221.

[14] Maluf, J.W. (2013), "The teleparallel equivalent of general relativity", *Annalen der Physik*, Vol. 525, pp. 339–357.

[15] Nester, J.M. (1991), "Normal frames for non-Riemannian connections", *Classical and Quantum Gravity*, Vol. 8, pp. L19–L23.

[16] Obukhov, Y.N. and Pereira, J.G. (2003), "Metric-affine approach to teleparallel gravity", *Physical Review D*, Vol. 67, Article ID 044016.

[17] Sciama, D.W. (1964), "The physical structure of general relativity", *Reviews of Modern Physics*, Vol. 36, pp. 463–469.

[18] Shapiro, I.L. (2002), "Physical aspects of the space-time torsion", *Physics Reports*, Vol. 357, pp. 113–213.

[19] Trautman, A. (2006), "Einstein–Cartan theory", in Françoise, J.-P., Naber, G.L. and Tsun, T.S. (Eds.), *Encyclopedia of Mathematical Physics*, Elsevier, pp. 189–195.

[20] Yaremenko, N. (2014), "Derivation of field equations in space with the geometric structure generated by metric and torsion", *Journal of Gravity*, Vol. 2014, Article ID 420123.